Numerical simulation of tidal bores and hydraulic jumps

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Abstract

An implicit finite difference formulation of the nonlinear shallow water equations is developed to allow for the treatment of tidal bores and hydraulic jumps. Five different schemes are investigated involving upwind treatment of convective terms, central differences combined with dissipative interface, forward time-centering and various combinations of these techniques. The schemes are analyzed with respect to their effective amplification portraits, and they are tested on periodic bores, uniform bores and steady hydraulic jumps. In this connection the model results are verified against analytical solutions and a numerical solution obtained with a Godunov Riemann solver. Scheme 4, which combines forward time centering and dissipative interface, is found to be superior to the others and it is applicable for Courant numbers within the range 0.25 to 1.5. This scheme is applied to a case study of the tidal bore in Huangzhou Bay and Qiantang River. The model results are shown to be in very good agreement with field data.

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1. Introduction

The main objective of this work has been to simulate the tidal bore in Huangzhou bay and Qiantang river south of Shanghai, China. During a visit to the Zhejiang Institute of Hydraulics and Estuary in 2002, the first author was inspired by this unique and fascinating natural wonder and he accepted the scientific challenge to develop the necessary modifications to the internationally well known and widely distributed commercial model MIKE 21. This resulted in a joint cooperation between the Technical University of Denmark (DTU), the Zhejiang Institute of Hydraulics and Estuary (ZIHE), Hangzhou, China and DHI-Water and Environment, Denmark.

This is not the first numerical study of the Qiantang bore, and previous investigations have been reported by e.g. Zhao (1985), Xin (1991), Tan et al. (1995), Su et al. (2001), Lin et al. (2002), Hui and Pan (2003), Pan et al. (2003), Su et al. (2001) established a 1D model covering the area from Ganpu to Fuchun power station, and a 2D model covering the local area from
Ganpu to Yanguan (see Fig. 7). They simulated a tidal event from August 1991 and compared with observed tidal elevations in the Qiantang River. Recently, 2D flexible grid models were established by Hui and Pan (2003) covering an area from Ganpu to Zakou, and by Pan et al. (2003) covering an area from Ganpu to Changqian. In both cases they calibrated the model using the extensive field campaign from September 2001.

Numerical methods for handling shocks and discontinuities have undergone a significant development over the last 20 years. An excellent review of shock-capturing methods is given by Toro (2001), who discuss advanced techniques such as the Godunov upwind scheme, the Godunov centred scheme, various approximate Riemann solvers and TVD methods such as the MUSCL-Hancock scheme. Such schemes have been applied in the previous studies of the Qiantang bore e.g. by Su et al. (2001), Hui and Pan (2003), Pan et al. (2003). Other recent and important examples are Hu et al. (1998) and Zhou et al. (2003).

Our ambition in the present work is more modest. We intend to modify an existing second order accurate implicit finite difference scheme (the S21, Jupiter scheme used in the commercial model MIKE 21), so that it becomes shock-capturing and can be used for tidal bores and steady hydraulic jumps. We do not claim that the resulting scheme will be as sophisticated as or more accurate than the Godunov and TVD schemes described by Toro (2001). However, the scheme will be extremely robust and efficient and it will allow relatively large time steps with Courant numbers up to say 2.

In Section 2, we investigate five different schemes involving upwind treatment of convective terms, central differences combined with dissipative interface, forward time-centering of the mass equation and some combinations of these techniques. The schemes are analyzed with respect to their amplification portraits. In Section 3, we test the five schemes on 3 canonical test cases: 1) The propagation of a periodic (tidal) bore; 2) The propagation and reflection of a uniform bore; and 3) The steady flow over a submerged bar involving a hydraulic jump. In Section 4, we use the recommended scheme to make a case study of the Qiantang tidal bore. Finally, summary and conclusions are made in Section 5.

2. Description of the numerical model

2.1. Introduction

The governing equations to be considered throughout this study are the depth-integrated Saint-Venant equations

\[
\frac{\partial \eta}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0, \tag{1}
\]

\[
\frac{\partial P}{\partial t} + gh \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial y} \left( \frac{P^2}{h} \right) + \frac{\partial}{\partial x} \left( \frac{PQ}{h} \right) + \tau_x = 0, \tag{2}
\]

\[
\frac{\partial Q}{\partial t} + gh \frac{\partial \eta}{\partial y} + \frac{\partial}{\partial x} \left( \frac{Q^2}{h} \right) + \frac{\partial}{\partial y} \left( \frac{PQ}{h} \right) + \tau_y = 0, \tag{3}
\]

where \(\eta\) is the surface elevation (measured from the still water level), \(h\) is the total water depth, and \((P,Q)\) are the depth-integrated flux components in the \(x\)- and \(y\)-directions, respectively. The bottom shear stress components are denoted \((\tau_x, \tau_y)\) and they are determined by

\[
(\tau_x, \tau_y) = \left( \frac{P}{h^2}, \frac{Q}{h} \right) \cdot \frac{g}{M^2 h^{1/3}} \sqrt{\frac{P^2 + Q^2}{h^2}}, \tag{4}
\]

where \(M\) is the Manning number with the dimensions \(m^{1/3}/s\) (the inverse of the commonly used \(n\)).

We emphasize that the governing differential Eqs. (1)–(3) conserve depth-integrated mass and momentum. This implies that continuous as well as discontinuous solutions are possible, and that the discontinuous solutions will satisfy the Rankine–Hugoniot jump conditions including the associated energy dissipation. On the other hand, the ability to model discontinuous solutions also heavily depends on the numerical scheme. Methods developed for continuous hyperbolic problems do not automatically work for discontinuous problems such as shocks, bores and hydraulic jumps. In fact, several aspects are likely to go wrong e.g. spurious, unphysical oscillations may occur in the vicinity of the discontinuity, and the strength and propagation speed of the shock may be severely off.

Our starting point in the present study has been the S21 Jupiter scheme, which was originally developed
for smooth continuous solutions. The basic properties of this scheme have been discussed in several publications (see e.g. Abbott et al., 1973, 1981; Abbott, 1979; Abbott and Basco, 1989), and during the last 30 years this scheme has been successfully applied in numerous studies involving hydrodynamics in coastal areas and estuaries. The scheme, which is described in Section 2.2, is based on a space staggered grid, finite difference approximations, and an alternating direction implicit (ADI) scheme. All the main terms in the equations are time- and space-centered, leading to second-order accuracy. Without modification this scheme is not suited for discontinuous problems.

Recently, McCowan et al. (2001) modified the S21 Jupiter scheme to deal with high Froude number flows in connection with floodplain applications. The most important modification was a shift from central to upwind treatment of the convective terms, which made it possible to simulate steady super-critical flows and for Courant numbers larger than 0.2. Consequently, with emphasis on modelling strong tidal bores, we have found it necessary to introduce further modifications to the S21 Jupiter scheme. These modifications are described in Sections 2.3–2.6.

2.2. The conventional second-order scheme (Scheme 0)

The starting point for our development is the S21-Jupiter scheme introduced by Abbott et al. (1973). In the following we provide a brief summary of this scheme (a detailed description can be found e.g. in Abbott et al., 1981). The scheme is formulated on a space staggered grid, using time- and space-centered finite difference approximations, and it is solved by an ADI method. First of all the mass equation is split into two parts, where the first part is combined with the x-momentum equation, while the second part is combined with the y-momentum equation.

In the x-sweep we solve for the unknowns $P^{n+1}$ and $h^{n+0.5}$, and the finite difference approximations used in Eqs. (1) and (2) read

$$\frac{\partial h}{\partial t} \approx \frac{2}{\Delta t} \left( \eta_{j,k}^{n+0.5} - \eta_{j,k}^n \right),$$

(5)

$$\frac{\partial P}{\partial x} \approx \frac{1}{2\Delta x} \left( P_{j,k}^{n+1} - P_{j-1,k}^{n+1} \right) + \frac{1}{2\Delta x} \left( P_{j,k}^n - P_{j-1,k}^n \right),$$

(6)

$$\frac{\partial Q}{\partial y} \approx \frac{1}{2\Delta y} \left( Q_{j,k}^{n+0.5} - Q_{j,k-1}^{n+0.5} \right) + \frac{1}{2\Delta y} \left( Q_{j,k}^{n-0.5} - Q_{j,k-1}^{n-0.5} \right),$$

(7)

$$\frac{\partial P}{\partial t} \approx \frac{1}{\Delta t} \left( P_{j,k}^{n+1} - P_{j,k}^n \right),$$

(8)

$$gh \frac{\partial \eta}{\partial x} = gh_{j+0.5,k}^{n+0.5} \frac{1}{\Delta x} \left( \eta_{j+0.5,k}^{n+0.5} - \eta_{j,k}^{n+0.5} \right),$$

(9)

$$\frac{\partial}{\partial x} \left( \frac{P^2}{h} \right) \approx \frac{1}{4\Delta x h_{j+1,k}^n} \left( P_{j+1,k}^{n+1} + P_{j,k}^{n+1} \right) \left( P_{j+1,k}^n + P_{j,k}^n \right) - \frac{1}{4\Delta x h_{j,k}^n} \left( P_{j,k}^{n+1} + P_{j-1,k}^{n+1} \right) \left( P_{j,k}^n + P_{j-1,k}^n \right),$$

(10)

$$\frac{\partial}{\partial y} \left( \frac{PQ}{h} \right) \approx \frac{1}{2\Delta y} \left( \left( P_{j+1,k}^n + P_{j,k}^n \right) v_1 - \left( P_{j,k}^n + P_{j-1,k}^n \right) v_2 \right),$$

(11)
where

\[ v_1 = \frac{2 \left( Q_{x,j,k}^{n+0.5} + Q_{x,j+1,k}^{n+0.5} \right)}{h_{j,k} + h_{j+1,k} + h_{j+1,k+1}}, \]  

and where \((a,b) = (n+1,n)\) during top-to-bottom sweeps (every odd sweep) and \((a,b) = (n,n+1)\) during bottom-to-top sweeps (every even sweep).

Similarly, in the \(y\)-sweep we solve for the unknowns \(Q_{y,j,k}^{n+1.5}\) and \(\eta_{y,j,k}^{n+1}\) and the finite difference approximations used in Eqs. (1) and (3) read

\[ \frac{\partial \eta}{\partial t} \approx \frac{2}{\Delta y} \left( \eta_{y,j,k}^{n+0.5} - \eta_{y,j,k}^{n} \right), \]

\[ \frac{\partial P}{\partial x} \approx \frac{1}{2\Delta x} \left( P_{x,j,k}^{n+1} - P_{x,j-1,k}^{n+1} \right) + \frac{1}{2\Delta x} \left( P_{x,j,k}^{n} - P_{x,j-1,k}^{n} \right), \]

\[ \frac{\partial Q}{\partial y} \approx \frac{1}{2\Delta y} \left( Q_{y,j,k}^{n+1.5} - Q_{y,j-1,k}^{n+1.5} \right) + \frac{1}{2\Delta y} \left( Q_{y,j,k}^{n+0.5} - Q_{y,j-1,k}^{n+0.5} \right), \]

\[ \frac{\partial Q}{\partial t} \approx \frac{1}{\Delta t} \left( Q_{y,j,k}^{n+1.5} - Q_{y,j,k}^{n+0.5} \right), \]

\[ gh \frac{\partial \eta}{\partial y} \approx gh_{y,j+0.5,k} \frac{1}{\Delta y} \left( \eta_{y,j+1,k}^{n+1} - \eta_{y,j,k}^{n+1} \right), \]

\[ \frac{\partial}{\partial y} \left( \frac{Q^2}{h} \right) \approx \frac{1}{4\Delta y h_{j+1,k+1}^{n+0.5}} \left( \left( Q_{y,j,k}^{n+1.5} + Q_{y,j,k}^{n+1.5} \right) \right) \]

\[ \times \left( Q_{x,j,k}^{n+0.5} + Q_{x,j,k}^{n+0.5} \right) \]

\[ - \frac{1}{4\Delta y h_{j+1,k+1}^{n+0.5}} \left( \left( Q_{y,j,k}^{n+1.5} + Q_{y,j,k-1}^{n+1.5} \right) \right) \]

\[ \times \left( Q_{x,j,k}^{n+0.5} + Q_{x,j,k-1}^{n+0.5} \right), \]

\[ \frac{\partial}{\partial x} \left( \frac{PQ}{h} \right) \approx \frac{1}{2\Delta x} \left( \left( Q_{x,j+1,k}^{n} + Q_{x,j,k}^{n} \right) v_{III} \right) \]

\[ - \left( Q_{x,j,k}^{n} + Q_{x,j-1,k}^{n} \right) v_{IV} \),

where

\[ v_\text{III} = \frac{2 \left( P_{x,j,k}^{n+1} + P_{x,j+1,k}^{n+1} \right)}{h_{j,k} + h_{j+1,k} + h_{j+1,k+1}}, \]

\[ v_\text{IV} = \frac{2 \left( P_{x,j-1,k}^{n+1} + P_{x,j,k}^{n+1} \right)}{h_{j-1,k} + h_{j,k} + h_{j+1,k+1}}, \]

and where \((a,b) = (n+1.5,n+0.5)\) during top-to-bottom sweeps (every odd sweep) and \((a,b) = (n+0.5,n+1.5)\) during bottom-to-top sweeps (every even sweep).

In a single horizontal dimension, the S21 Jupiter scheme simplifies in the following way: By discarding e.g. the \(y\)-momentum equation, the \(y\)-mass equation becomes an explicit dummy operation which extrapolates, \(\eta\) from \(n+0.5\) to \(n+1\), and as a consequence it can be shown that

\[ \eta_{y,j,k}^{n+0.5} = \frac{1}{2} \left( \eta_{y,j,k}^{n} + \eta_{y,j,k}^{n+1} \right). \]

By inserting this result into Eq. (5) we get

\[ \frac{\partial \eta}{\partial t} \approx \frac{1}{\Delta t} \left( \eta_{y,j,k}^{n+1} - \eta_{y,j,k}^{n} \right), \]

which should be combined with Eq. (6). Similarly, by inserting Eq. (23) in Eq. (9) we get

\[ gh \frac{\partial \eta}{\partial x} \approx gh_{y,j+0.5,k} \frac{1}{\Delta x} \left( \eta_{y,j+1,k}^{n+1} - \eta_{y,j,k}^{n+1} \right) \]

\[ + \frac{1}{\Delta x} \left( \eta_{y,j+1,k}^{n} - \eta_{y,j,k}^{n} \right) \]

which should be combined with Eqs. (8) and (10).

2.3. Selective up-winding of the convective momentum terms (Scheme 1)

In regions where steep velocity gradients occur, central differences used in combination with the nonlinear convective terms may lead to spurious oscillations. This makes it impossible to apply the conventional second-order scheme for supercritical flow and tidal bores. To solve this problem, McCowan et al. (2001) recently suggested to modify the scheme by using a selective up-winding of the convective
terms in the direction from which the flow is coming. Hence they replaced Eqs. (10) and (19) by
\[
\frac{\partial}{\partial x} \left( \frac{P^2}{h} \right)_{\text{upwind}} \approx \frac{1}{\Delta x h_{j,k}^n} \left( P_{j,k}^{n+1} - P_{j-1,k}^{n+1} P_{j-1,k}^n \right),
\] (26)

\[
\frac{\partial}{\partial y} \left( \frac{Q^2}{h} \right)_{\text{upwind}} \approx \frac{1}{\Delta y h_{j,k}^{n+0.5}} \times \left( Q_{j,k}^{n+1.5} Q_{j,k}^{n+0.5} - Q_{j,k-1}^{n+1.5} Q_{j,k-1}^{n+0.5} \right),
\] (27)

while the cross-momentum terms were treated by central-differences as described by Eqs. (11) and (20).

By the use of Taylor series expansions we can show that
\[
\frac{\partial}{\partial x} \left( \frac{P^2}{h} \right)_{\text{upwind}} \approx \frac{\partial}{\partial x} \left( \frac{P^2}{h} \right)_{\text{central}} - \frac{\Delta x}{2} \frac{\partial^2}{\partial x^2} \left( \frac{P^2}{h} \right),
\] (28)

hence the up-winding technique corresponds to introducing a first-order dissipative term. This term will tend to damp oscillations with high wave numbers (poorly resolved oscillations), while it will have little effect on low wave numbers (well-resolved phenomena).

To ensure that the dissipative effect of the up-winding was only included when necessary, McCowan et al. (2001) introduced a switch between the upwind- and central-difference representations
\[
\frac{\partial}{\partial x} \left( \frac{P^2}{h} \right) = \alpha \frac{\partial}{\partial x} \left( \frac{P^2}{h} \right)_{\text{upwind}} + (1 - \alpha) \frac{\partial}{\partial x} \left( \frac{P^2}{h} \right)_{\text{central}},
\] (29)

where \( \alpha \) was a function of the Eulerian Froude Number \( (F_R = U/\sqrt{gh}) \). The weighting factor \( \alpha \) was calculated for each grid point at each time step, immediately prior to the calculation of the momentum equation coefficients. This ensured that numerical dissipation was only introduced at grid points where high Froude number flow was occurring, and that the normal high accuracy solution was obtained throughout the rest of the model domain. McCowan et al. (2001) demonstrated that their procedure worked efficiently on steady phenomena involving e.g. the transition from super-critical flow to subcritical flow.

Unfortunately, there are evident problems with this method for unsteady flow. First of all, the detection based on an Eulerian determination of the Froude number completely fails in the case of tidal bores, where the magnitude and direction of the particle velocity do not define the magnitude or the direction of the travelling bore (see e.g. Section 3.2). Secondly, it turns out that the up-winding technique is only efficient for Courant numbers less than say 0.1, beyond which spurious oscillations will not be suppressed (see Sections 3.1 and 3.2). This is a disappointing limitation for an implicit scheme. Thirdly, in its present implementation where cross-momentum terms are treated by central differences, the method will only be efficient if the flow is in the \( x \)- or \( y \)-direction, and generally spurious oscillations may arise in e.g. the diagonal grid direction.

### 2.4. Forward time-centering of the spatial derivatives (Scheme 2)

As discussed in the previous subsections, spurious oscillations occurring at the front of a moving tidal bore can be removed by making the scheme dissipative for the poorly resolved wave numbers. One option is to introduce a fully implicit treatment of the spatial derivatives, which become forward centered instead of mid-centered in time. In a single horizontal dimension this is possible in the mass equation as well as in the momentum equation and leads to a replacement of Eq. (6) by
\[
\frac{\partial P}{\partial x} = \frac{1}{\Delta x} \left( P_{j,k}^{n+1} - P_{j-1,k}^{n+1} \right),
\] (30)

and a replacement of Eq. (25) by
\[
gh \frac{\partial h}{\partial x} = gh_{j+0.5,k} \frac{1}{\Delta x} \left( h_{j+1,k}^{n+1} - h_{j,k}^{n+1} \right).
\] (31)

With this modification the 1D descent of the S21 Jupiter scheme becomes identical to the Euler implicit scheme (see e.g. Hirsch, 1988, his Section 10.3.3).

In the two-dimensional version of the S21-Jupiter scheme it is, however, not possible to forward time-center the gravity terms in the momentum equations, as they are already being evaluated at the newest time-
level according to Eqs. (9) and (18). Therefore, spatial
derivatives will only be forward time-centered in the
two mass equations. Consequently, in the x-sweep we
replace Eq. (6) by Eq. (30) and Eq. (7) by
\[
\frac{\partial Q}{\partial y} \approx \frac{1}{\Delta y} \left( Q_{y,n+0.5}^x - Q_{y,n-0.5}^x \right),
\]
while in the y-sweep we replace Eqs. (15) and (16) by
\[
\frac{\partial P}{\partial x} \approx \frac{1}{\Delta x} \left( P_{x, n+1}^y - P_{x, n-1}^y \right),
\]
and
\[
\frac{\partial Q}{\partial y} \approx \frac{1}{\Delta y} \left( Q_{y,n+1.5}^x - Q_{y,n-1.5}^x \right),
\]
respectively. This modification of the mass equations
will introduce a dissipative effect, which will be
analyzed in the following.

In order to analyze the effect of the modified S21
Jupiter scheme we make a linear Fourier analysis of the
1D equations, with and without the forward
centering of the spatial derivatives in the mass
equation. We can express this problem in the matrix
form
\[
\begin{pmatrix}
\frac{1}{\Delta x} & \frac{1}{2} \frac{(2-\delta)\Delta t \Omega}{gh} & \frac{1}{\Delta x} & \frac{1}{2} \frac{(2-\delta)\Delta t \Omega}{gh} \\
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\Delta x} & \frac{1}{2} \frac{(2-\delta)\Delta t \Omega}{gh} & \frac{1}{\Delta x} & \frac{1}{2} \frac{(2-\delta)\Delta t \Omega}{gh} \\
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2 \\
\eta_3 \\
\eta_4 \\
\end{pmatrix}^{n+1}
\begin{pmatrix}
\eta_1 \\
\eta_2 \\
\eta_3 \\
\eta_4 \\
\end{pmatrix}^n,
\]
where \(\delta=1\) represents the original mid-centering
(Scheme 0), and \(\delta=0\) represents full forward centering
of the spatial derivative of \(P\) in the mass equation
(Scheme 2). Note that \(\Omega\) accounts for the discrete
representation of the spatial derivative of \(P\) and that a
central difference on a space-staggered grid yields
\[
\Omega = i \frac{2}{\Delta x} \sin \left( \frac{\pi}{N_x} \right), \quad N_x = 2, 3, \ldots, \infty
\]
where \(N_x\) is the number of grid points per wave
length and \(i\) is the imaginary number. On the basis of Eq.
(35) we can now determine the amplification matrix
\[
G = \begin{pmatrix}
\frac{1}{\Delta x} & \frac{1}{2} \frac{(2-\delta)\Delta t \Omega}{gh} & \frac{1}{\Delta x} & \frac{1}{2} \frac{(2-\delta)\Delta t \Omega}{gh} \\
\end{pmatrix}^{-1}
\begin{pmatrix}
\frac{1}{\Delta x} & \frac{1}{2} \frac{(2-\delta)\Delta t \Omega}{gh} & \frac{1}{\Delta x} & \frac{1}{2} \frac{(2-\delta)\Delta t \Omega}{gh} \\
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2 \\
\eta_3 \\
\eta_4 \\
\end{pmatrix}^{n+1}
\begin{pmatrix}
\eta_1 \\
\eta_2 \\
\eta_3 \\
\eta_4 \\
\end{pmatrix}^n,
\]
having the eigenvalues
\[
\lambda = \frac{\Phi^2 - 1}{\Phi^2} \pm \Phi \sqrt{\Phi^2(1-\delta)^2 - 4},
\]
where
\[
\Phi = C \sin \left( \frac{\pi}{N_x} \right), \quad C_r = \frac{\Delta t}{\Delta x} \sqrt{gh}.
\]

Fig. 1 shows the trace of eigenvalues in the
complex plane for varying values of \(\Phi\). It is plotted
for the cases of \(\delta=1\) (Scheme 0) and \(\delta=0\) (Scheme
2) both with \(C_r=3\). With \(N_x\) covering all possible
wave numbers from Nyquist (i.e. \(N_x=2\)) to infinity,
the variable \(\Phi\) will cover the interval from \(C_r\) to
zero. For the case of \(\delta=1\) (i.e. the conventional
second order scheme) all eigenvalues will be located
on the unit circle, i.e. the scheme is neutrally stable
without dissipation. For \(1>\delta\geq0\) (i.e. forward
time-centering) the eigenvalues can belong to two different
regimes:

I) If \(\Phi^2(1-\delta)^2<4\) (regime 1) the eigenvalues are
complex and will be located on a circle which is centred on the real axis at the abscissa \((1-\delta)/(3-\delta)\)
and having the radius of \(2/(3-\delta)\). This is inside or on the unit circle and indicates a stable but dissipative scheme. In this case the amplification factor (the absolute value of the maximum eigenvalue) will have a minimum for \(N_x=2\) i.e. for \(\Phi=C_r\), and it will steadily grow towards unity for increasing \(N_x\).

II) If \(\Phi^2(1-\delta)^2\geq4\) (regime II) the eigenvalues will be located on the negative real axis. Note that \(\Phi^2(1-\delta)^2=4\) may be satisfied for a certain value of \(N_x^*\) if \(C r^2(1-\delta)^2 \geq 4\). In this case \(N_x>N_x^*\) leads to eigenvalues in regime I, while \(N_x=N_x^*\) makes the two eigenvalues from (38) coincide at the abscissa \((-1-\delta)/(3-\delta)\). It turns out that for \(2 \leq N_x \leq N_x^*\) one eigenvalue moves towards \(-1\) while the other moves towards zero. As a consequence, maximum dissipation will occur at \(N_x=N_x^*\), while shorter disturbances will be lees damped.

The amplification factor is usually defined as the absolute value of the maximum eigenvalue of the
amplification matrix. This corresponds to the dissipation rate per time step. However, to investigate the dissipation accumulated over a given time span, we must take into account that a reduction in the Courant number leads to an equivalent increase in the number of time steps. We do this by introducing the measure $K_{u,j}$:

$$K_{u,j} = \frac{\text{Cr}_j}{2}.$$  \hfill (40)

Curves 2a and 2b in Fig. 2 show $A$ as a function of $N_x$ for Scheme 2 using $Cr=0.25$ and $Cr=2.0$, respectively. We notice that Scheme 2 becomes increasingly dissipative for larger Courant numbers.

The forward time-centering technique is well suited for unsteady problems involving rapid spurious oscillations. It is also effective during the transient phase of an iteration towards a steady state, where rather large time steps can be applied in a hyperbolic formulation. On the other hand, this technique cannot prevent spurious oscillations in a stationary solution which contains large gradients (such as the hydraulic jump discussed in Section 3.3). It is therefore relevant to combine it with either the upwind method described in the previous section or with the explicit dissipative method dimmed in the following section.

2.5. Dissipative interface and forward time-centering (Schemes 3 and 4)

A classical way of avoiding spurious oscillations in the numerical solution is to introduce an explicit smoothing filter (often denoted as a dissipative interface) on one or both of the governing variables e.g.

$$\eta_j^* = \beta \eta_{j-1} + (1 - 2\beta) \eta_j + \beta \eta_{j+1}, \quad 0 \leq \beta \leq 0.5. \hfill (41)$$

In Fourier space the representation of this operation reads

$$\eta_j^* = \gamma \eta_j, \hfill (42)$$

where

$$\gamma = 1 + 2\beta \left( \cos \left( \frac{2\pi}{N_x} \right) - 1 \right). \hfill (43)$$
Hence, if we perform this filtering after each time step on $\eta$ as well as on $P$, it corresponds to multiplying the previously established amplification matrix (Eq. (37)) with $c_0$:

$$\frac{U^2}{c_0^2} \sqrt{\Phi^2 (1 - \delta)^2 - 4}$$

This leads to the eigenvalues

$$\lambda = \gamma \left( \frac{\Phi^2 - 1}{\Phi^2 (\delta - 2) - 1} \right)^{\frac{1}{2}}$$

where $\Phi$ is defined by Eq. (39). This scheme will be stable and dissipative for $0 \leq \delta < 1$ and $0 < \beta \leq 0.5$. We note that $(\delta, \beta) = (1, 0)$ leads to Scheme 0 from Section 2.2, while $(\delta, \beta) = (0, 0)$ leads to Scheme 2 from Section 2.4. We define Scheme 3 by $(\delta = 1, \beta = 0.25)$ i.e. mid-centering combined with dissipative interface, and Scheme 4 by $(\delta = 0, \beta = 0.25)$ i.e. forward time-centering combined with dissipative interface.

The accumulated amplification factor (Eq. (40)) is shown in Fig. 2, where curves 3a and 3b represent Scheme 3 (for the Courant numbers $Cr = 0.25$ and $Cr = 2.0$, respectively), while curves 4a and 4b are similar representations of Scheme 4. First of all, we notice that Scheme 3 becomes increasingly dissipative for smaller Courant numbers while it is less efficient for Courant numbers above 1. This is in contrast to Scheme 2. Scheme 4 combines the two methods and as a result it becomes strongly dissipative for large as well as for small Courant numbers. This is attractive as we can use it for shock-capturing within a flexible interval of Courant numbers, say $0.25 \leq Cr \leq 2.0$. Further testing of these schemes will be made in Section 3.

2.6. Relaxation techniques at open boundaries

The problem of weakly reflective open boundary conditions have received considerable attention within the last 30 years, and some of the pioneers were Orlanaki (1976), Engquist and Majda (1977), Israeli and Orzag (1981), and Larsen and Dancy (1983). It is an essential problem in connection with wave modelling, where reflections from breakwaters, coasts or simply uneven topography may return to the incoming boundary. Unless this boundary is treated in a special way, these waves will be re-reflected back into the model area and falsify the solution.

The problem is generally less severe in connection with tidal modelling although storm surges may create a significant amount of reflection. However, the special case of tidal bores, which we focus on in this work, can be considered as a combination of conventional tidal flow and strongly nonlinear waves. In the case study described in Section 4, the ocean boundary...
is a 12 h sinusoidal tidal elevation, while the up-river boundary condition includes a strong bore forcing the water level to increase several meters within 15–30 min. If the numerical bore approaching this boundary has only a slight phase shift compared to the measured bore, this will create a conflict at the up-river boundary and result in the generation of a strong reflection.

As discussed by e.g. Jensen (1998), a simple but efficient method to reduce reflections from the boundary is to apply the relaxation

\[ \Phi^*(x, y, t) = \kappa(x, y) \Phi_{BC}(t) + (1 - \kappa(x, y)) \Phi(x, y, t), \]

where \( \Phi_{BC} \) is the measured time series, \( \Phi \) is the computed solution (before relaxation is applied) and \( \Phi^* \) is the corrected solution. This relaxation is performed on the surface elevation as well as on the flux or velocity variable, and it is applied as an explicit correction after each time step of the computation. The key to a successful reduction of reflections is the variation of the parameter \( k(x, y) \). This should take the value of unity at the boundary and zero at the end of the relaxation zone, and its variation should be rapid close to the boundary and extremely mild at the end of the zone. Many possible functions can be designed, and one suggestion is to use

\[ \kappa(\sigma) = \left( 1 - \frac{3}{5} \sigma \right)^8, \quad 0 \leq \sigma \leq 1, \]

where \( \sigma \) is a scaled coordinate perpendicular to the boundary with the value \( \sigma = 0 \) at the boundary and \( \sigma = 1 \) at the end of the relaxation zone. The extent of the relaxation zone is also an important parameter. For wave problems we typically recommend at least half a wave length, but this is of course not possible for tidal waves. For our particular purpose of tidal bores it turns out that the layer should be wide enough to cover the phase lag between the computed bore and the measured bore i.e. the outgoing bore should at least reach the outskirts of the relaxation zone by the time it appears in the measurements.

3. Verification of methods on idealized test cases

In the previous sections we have described a number of possible adjustments which can be made to the existing second order S21 scheme in order to improve its capabilities with respect to bores and hydraulic jumps. The following schemes have been implemented in the framework of S21:

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Original S21 Jupiter scheme</td>
</tr>
<tr>
<td>1</td>
<td>Upwind of convective terms</td>
</tr>
<tr>
<td>2</td>
<td>Forward time-centering of mass equation ((\delta=0, \beta=0))</td>
</tr>
<tr>
<td>3</td>
<td>Dissipative interface and central differences ((\delta=1, \beta=0.25))</td>
</tr>
<tr>
<td>4</td>
<td>Forward time-centering+Dissipative interface ((\delta=0, \beta=0.25))</td>
</tr>
<tr>
<td>5</td>
<td>Forward time-centering+Upwind of convective terms ((\delta=0))</td>
</tr>
</tbody>
</table>

The stability analyses and the resulting amplification portraits made in the previous sections do not guarantee that the methods perform in the case of nonlinear problems such tidal bores and hydraulic jumps. Hence, in this section we model three different idealized test cases with the objective of emphasizing the limitations of each of the methods listed above. The three test cases include a) the propagation of a highly nonlinear tidal wave which develops into a periodic bore; b) the propagation and reflection of a uniform bore; and c) the formation of a stationary hydraulic jump over a submerged bar.

3.1. Example 1: development and propagation of a periodic bore

In this example we study the development and propagation of a periodic bore over a constant depth. A linear sinusoidal wave is generated at the western boundary but due to nonlinearity and amplitude dispersion this wave will quickly become steeper until it forms an almost vertical front face i.e. a bore. During the early stages of the bore development, a close up of the wave front resembles a uniform bore but gradually the surface profile transforms into a triangular shape with a steadily decreasing wave height (see Whitham, 1974). This example is an idealised study of tidal bores, which typically develop due to topographical focusing effects, and it is a relevant simplification of the Qiantang bore, which is studied in detail in Section 4.

At first, we study the early developments of the steep front of the bore by applying the schemes
Fig. 3. a. Development of the front of a periodic tidal bore. Time series of surface elevations at location \(x=200\) km. \(C_r=1.88\). b. As (a) but with \(C_r=0.95\). c. As (a) but with \(C_r=0.31\). d. As (a) but with \(C_r=0.06\).
discussed in Section 2. We compare these results to a reference solution obtained by the highly accurate Godunov scheme of Pan et al. (2003). We consider a 1D channel with a length of 500 km, and a constant depth of $h=4.0$ m. At the western boundary, the surface elevation is

$$\eta_{\text{west}}(t) = A \sin(\omega t), \quad \omega = \frac{2\pi}{T},$$

(47)

where $A=2.0$ m, and $T=12$ h. The eastern boundary is absorbing so that reflection is not an issue in this test case. Bottom friction is ignored in the simulations. The grid size is $\Delta x=100$ m and simulations are made with time steps of $\Delta t=1$ s, 5 s, 10 s, 15 s and 30 s leading to the Courant numbers $Cr=0.06, 0.31, 0.62, 0.95$ and 1.88, respectively. Note that these Courant numbers are defined in terms of the linear shallow water celerity using the initial depth $h=4.0$ m. The reference solution by Pan et al. (2003) is made with $Cr=0.31$.

Fig. 3a shows the case of $Cr=1.88$ and we notice that Schemes 4 and 5 are superior while Scheme 2 is almost of the same quality. These three schemes have forward time-centering as a common feature. In contrast, Schemes 1 and 3 both fail to remove spurious oscillations and they lead to a significant phase lag. Fig. 3b shows the case of $Cr=0.95$, and reveals that Scheme 4 is the best, followed by Schemes 5, 2 and 3. Again Scheme 1 fails and leads to a significant phase lag. For the case of $Cr=0.63$ (not shown) Scheme 4 is again the best, followed by Schemes 3, 5, 2 and 1. Fig. 3c shows the case of $Cr=0.31$, and reveals that Schemes 4 and 3 are superior, followed by Scheme 5 which has a steeper front but is somewhat delayed. Schemes 1 and 2 are lagging behind and contain some oscillations. Finally, for the case of $Cr=0.06$ (Fig. 3d) Schemes 1 and 5 become superior in steepness as well as in speed. Now Scheme 2 does not suppress spurious oscillations, while Schemes 3 and 4 are over-diffusive with a smeared out front.

On this basis, it can be concluded that only for very low Courant numbers, the upwind technique in Scheme 1 performs well. It can also be concluded that the use of dissipative interface works well in the interval $0.3 \leq Cr \leq 0.95$, but should not be applied for smaller Courant numbers where it completely smears out the front. In contrast, the technique of forward time-centering works well for $Cr \geq 0.95$, while it has limited effect for smaller Courant numbers. Scheme 4 combines the forward time centering (from Scheme 2) and the dissipative interface (from Scheme 3) and for a typical range of Courant numbers such as $0.3 \leq Cr \leq 2$, this scheme is the most accurate and robust of the five tested schemes. However, also Scheme 5 appears to be an attractive scheme, and it has the advantage that it will cover the region $0 < Cr \leq 2$, without smearing out the solution.
Next, we study the long term development of the periodic bore from the initial sinusoidal shape to the asymptotic triangular shape. For this purpose we extend the dimension of the channel to 2400 km with a grid size of \( \Delta x = 200 \) m. Scheme 4 is executed with a time step of \( \Delta t = 20 \) s (i.e. \( Cr = 0.62 \)), while the Godunov model of Pan et al. (2003) is applied with \( \Delta t = 3 \) s (higher time steps make the Godunov solution oscillate). At the west boundary we apply a sinusoidal flux condition with amplitude \( Q_0 = 5.0 \) m\(^2\) s\(^{-1}\). Fig. 4a shows the spatial variation of the surface elevation after 100 h (i.e. 18,000 time steps of 20 s). We note that the solutions obtained with Scheme 4 and with the Godunov solver are very similar in height as well as in phase. This is also confirmed by the close up shown in Fig. 4b. The triangular shape of the profile is very distinct, and we find that the asymptotic development follows the description of Whitham (1974): a) The bore speed (and the wave length) becomes practically constant after 350 km and it is found to be \( c = 6.34 \) m/s in fairly good agreement with the linear celerity based on the still water depth;

![Graph showing surface elevation](image_url)

Fig. 4. a. Long term development of a periodic tidal bore. Spatial variation of the surface elevation at time \( t = 100 \) h. Dashed line: Scheme 4 with \( \Delta t = 20 \) s. Full line: Pan et al. (2003) with \( \Delta t = 3 \) s. b. Close up of (a). Dashed line: Scheme 4 with \( \Delta t = 20 \) s. Full line: Pan et al. (2003) with \( \Delta t = 3 \) s.
Table 1
Propagating bore

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$Cr$</th>
<th>Scheme 1</th>
<th>Scheme 2</th>
<th>Scheme 3</th>
<th>Scheme 4</th>
<th>Scheme 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_B$</td>
<td>$c$</td>
<td>$d_B$</td>
<td>$c$</td>
<td>$d_B$</td>
<td>$c$</td>
<td>$d_B$</td>
</tr>
<tr>
<td>0.05</td>
<td>0.16</td>
<td>2.54</td>
<td>6.50</td>
<td>2.58</td>
<td>6.28</td>
<td>2.51</td>
</tr>
<tr>
<td>0.10</td>
<td>0.31</td>
<td>2.56</td>
<td>6.42</td>
<td>2.61</td>
<td>6.20</td>
<td>2.53</td>
</tr>
<tr>
<td>0.25</td>
<td>0.78</td>
<td>2.66</td>
<td>6.05</td>
<td>2.68</td>
<td>5.93</td>
<td>2.61</td>
</tr>
<tr>
<td>0.50</td>
<td>1.57</td>
<td>2.80</td>
<td>5.56</td>
<td>2.72</td>
<td>5.82</td>
<td>2.75</td>
</tr>
<tr>
<td>1.0</td>
<td>3.14</td>
<td>2.90</td>
<td>5.32</td>
<td>2.73</td>
<td>5.79</td>
<td>2.87</td>
</tr>
</tbody>
</table>

In addition to Eqs. (49) and (50) mass conservation at the western boundary yields

$$u_B d_B = Q_0.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (51)

For given values of $Q_0$, $d_A$ and $u_A$ we can determine $c$, $d_B$ and $u_B$ by solving Eqs. (49–51). This will test the shock-capturing capabilities of the numerical schemes, and only if they result in the proper energy burnout at the bore front, the speed and the height of the bore will be correctly modelled.

After the bore has reflected against the vertical wall, we can again apply Eqs. (49–51) with a few modifications: At the western boundary we still denote conditions by $d_B$ and $u_B$ and these are identical to the solution just found. At the eastern boundary and behind the reflected bore the unknown new depth is now denoted $d^*_B$ while the corresponding velocity $u^*_B$ becomes zero in agreement with the wall conditions. Based on this input we can solve the above equations with respect to $c^*$ and $d^*_B$. As an example we consider the input: $Q_0=10$ m$^2$/s and $d_A=1.0$ m, which leads to the analytical solutions:

**Propagating bore**: $d_B = 2.517$ m, $c = 6.59$ m/s,

**Reflected bore**:

$x_A^* = 4.824$ m, $c^* = -4.33$ m/s.

Numerical solutions are computed with Schemes 1–5 using a grid size of $\Delta x = 1$ m and time steps of $\Delta t = 0.05$ s, 0.10 s, 0.25 s, 0.50 s and 1.0 s. We summarize the results in Tables 1 and 2. For simplicity, we define the Courant numbers in terms of the linear shallow water celerity using the initial depth $h=1.0$ m. The actual speed of the bore is more than twice this celerity.

On this basis the following conclusion can be made: Scheme 3 perform relatively well for Courant

Table 2
Reflected bore

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$Cr$</th>
<th>Scheme 2</th>
<th>Scheme 3</th>
<th>Scheme 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d^*_B$</td>
<td>$c^*$</td>
<td>$d^*_B$</td>
<td>$c^*$</td>
<td>$d^*_B$</td>
</tr>
<tr>
<td>0.05</td>
<td>0.16</td>
<td>4.91</td>
<td>-4.29</td>
<td>4.83</td>
</tr>
<tr>
<td>0.10</td>
<td>0.31</td>
<td>4.96</td>
<td>-4.27</td>
<td>4.86</td>
</tr>
<tr>
<td>0.25</td>
<td>0.78</td>
<td>5.03</td>
<td>-4.27</td>
<td>5.00</td>
</tr>
<tr>
<td>0.50</td>
<td>1.57</td>
<td>5.05</td>
<td>-4.29</td>
<td>5.26</td>
</tr>
<tr>
<td>1.0</td>
<td>3.14</td>
<td>5.06</td>
<td>-4.30</td>
<td>5.56</td>
</tr>
</tbody>
</table>

b) The wave height decreases with time and distance approximately following the decay of $1/t$ or $1/t$ as predicted by Whitham (1974).

3.2. Example 2: propagation and reflection of a uniform bore

In this example we study the propagation and reflection of a uniform bore moving over a horizontal bottom. Analytical solutions are easily derived and the test case has previously been described by e.g. Abbott (1979), in his Chapter 1. We consider a 1D channel of constant depth $d_A$ and with initially calm water i.e. $u_A=0$. The eastern boundary is closed (i.e. fully reflective) while, at the western boundary, we impose a Heaviside step function and describe the discharge per unit width by

$$q(t) = Q_0 \text{ m}^2/\text{s} \quad \text{for} \ t \geq 0.$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (48)

This will provoke a moving uniform bore with a very steep front. Behind the bore the depth will increase to $d_B$ and the velocity to $u_B$, while the speed of this bore will be $c$. The phenomenon is steady from a coordinate system moving with the bore and by using conservation of mass and momentum we obtain the conditions

$$(c - u_B)d_B = (c - u_A)d_A,$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (49)

$$(c - u_B)^2 + \frac{1}{2} gd_B^2 = d_A(c - u_A)^2 + \frac{1}{2} gd_A^2.$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (50)

We note that an alternative to Eq. (50) can be obtained by eliminating $c - u_B$ by the use of Eq. (49) which leads to

$$(c - u_A)^2 = g \frac{d_B}{d_A} \left( \frac{d_A + d_B}{2} \right).$$
numbers up to 1, while Scheme 2 is better for larger Courant numbers. Scheme 1 is only acceptable for $Cr \leq 0.16$, and it completely fails when the bore is reflected from the eastern boundary. The reason is that the imbedded switch for the upwinding scheme (as described in Section 2.3) is a function of the Eulerian particle velocity, and this switch becomes useless when the velocity goes to zero behind the reflected bore. Scheme 5 is a combination of Schemes 1 and 2, and consequently has the same limitation for reflected bores as Scheme 1. However, the scheme is quite accurate for the propagating bore and it is the best performing scheme for small Courant numbers.

Finally, we conclude from Tables 1 and 2 that Scheme 4 leads to the most accurate determination of the bore height and bore speed and that it has a fairly good accuracy for a range of Courant numbers $0.16 \leq Cr \leq 1.57$. This is supported by Fig. 5a,b, which show the computed spatial variation of the surface elevation at time $t=25$ s (propagating mode) and $t=60$ s (reflected mode) compared to the analytical solution. The agreement is most acceptable for all the shown Courant numbers. Add to this the conclusion from

---

Fig. 5. a. Propagation of a uniform bore on a constant depth. Spatial variation of the surface elevation at time $t=25$ s. Scheme 4 versus analytical solution. 1: $Cr=0.16$; 2: $Cr=0.31$; 3: $Cr=0.78$; 4: $Cr=1.57$. b. Reflection of a uniform bore on a constant depth. Spatial variation of the surface elevation at time $t=60$ s. Scheme 4 versus analytical solution. 1: $Cr=0.16$; 2: $Cr=0.31$; 3: $Cr=0.78$; 4: $Cr=1.57$. 
these previous subsections and it becomes evident that Scheme 4 is the best of the tested schemes. Consequently, this is the scheme we recommend and it is our preference to be used throughout the rest of this paper.

3.3. Example 3: steady hydraulic jump over a bar

Critical flow over a submerged bar is a well known problem from classical text books (see e.g. Chow, 1959) and this case has previously been used to verify numerical shock-capturing schemes (see e.g. Zhou et al., 2001). An excellent discussion of the general physics involved can be found in Houghton and Kasahara (1968), who considered the related case where a submerged bar is impulsively given a constant horizontal velocity in an infinite domain of calm water.

In the following we shall consider the simpler case of a fixed bar with the geometry (measured from the otherwise horizontal bottom) defined by

\[
\Gamma(x) = \Gamma_c \left(1 - \left(\frac{x-x_e}{b}\right)^2\right), \quad x_e - b \leq x \leq x_e + b.
\]

(52)

Initially the water is assumed to be at rest with a depth of \(d_0\) in the horizontal regions of the domain. At \(t=0\) we impulsively turn on a constant flux \(Q_0\) m\(^2\)/s at the western (upstream) boundary, and this will generate a moving uniform bore as studied in example 2. This bore will be partially reflected and partially transmitted over the bar and later it will bounce off against the eastern boundary. However, after a while the flow will settle down and approach an asymptotic steady state which may involve a combination of subcritical flow and supercritical flow, and it may also involve a stationary hydraulic jump located somewhere on the bar. The actual type of the asymptotic flow depends on the combination of parameters \(Q_0\), \(d_0\) and \(\Gamma_c\) and on the eastern (downstream) boundary condition.

We can characterize the flow pattern over the bar, by first assuming a continuous (energy conserving) steady state throughout the model area, which leads to the three conditions

\[
\frac{u(x)^2}{2g} + d(x) + \Gamma(x) = \frac{u_0^2}{2g} + d_0,
\]

(53)

\[
u(x)d(x) = u_0d_0 = Q_0.
\]

(54)

By inserting Eq. (54) in Eq. (53) we obtain

\[
\frac{F_0^2}{2} U^3(x) + \left(M(x) - \frac{F_0^2}{2} - 1\right) U(x) + 1 = 0,
\]

(55)

where

\[
F_0 = \frac{u_0}{\sqrt{g d_0}}, \quad U(x) = \frac{u(x)}{u_0},
\]

\[
M(x) = \frac{\Gamma(x)}{d_0}, \quad M_c = \frac{\Gamma_c}{d_0}.
\]

(56)

Next, we solve Eq. (55) with respect to \(M(x)\) as a function of \(U(x)\). Only \(U>0\) and \(M>0\) are of interest, and for a given Froudes number \(F_0\) this defines a maximum \(M^*\) beyond which no continuous flow solution exists. This maximum is found where \(\partial M/\partial U=0\) and it leads to

\[
M^* = 1 + \frac{F_0^2}{2} - 3\frac{F_0^{2/3}}{2}, \quad U^* = F_0^{-2/3},
\]

(57)

which corresponds to critical flow conditions i.e. \(F^* = 1\). The magnitude of \(M_c\) relative to the limiting \(M^*\) defines the different possible flow regimes over the bar: In Regime I we have \(F_0<1\) and \(M_c<M^*\) and this is characterized by subcritical flow throughout the domain and a symmetrical dip in the surface elevation over the bar. In Regime III we have \(F_0>1\) and \(M_c<M^*\) and this is characterized by supercritical flow throughout the domain and a symmetrical rise in the surface elevation over the bar.

Our primary interest here is Regime II, which is initially characterized by \(M_c>M^*\) and in this case the initial water depth \(d_0\) cannot be sustained at the upstream boundary. Now the bar will partially block the flow and the upstream depth will gradually change from \(d_0\) to \(d_\Lambda\) (and \(F_0\) to \(F_\Lambda\)) so that critical flow conditions are eventually reached on the crest of the bar. The asymptotic conditions at the upstream boundary \((d_\Lambda\) and \(u_\Lambda\)) and at the bar crest \((d_c\) and \(u_c\)) are determined from conservation of energy and mass combined with the critical flow condition i.e. the four equations

\[
\frac{u_c^2}{2g} + d_c + \Gamma_c = \frac{u_\Lambda^2}{2g} + d_\Lambda, \quad u_c d_c = u_\Lambda d_\Lambda = Q_0,
\]

\[
u_c = \sqrt{gd_c}.
\]

(58)
Downstream of the bar crest the flow will be supercritical which may persist all the way to the downstream boundary unless a particular depth $d_B$ is enforced there. If that happens the flow will have to return to a subcritical state through a hydraulic jump, which typically will be located somewhere on the downstream face of the bar. The unknowns in this problem are $d_{\text{toe}}$, $u_{\text{toe}}$ and $\Gamma_{\text{toe}}$ at the toe of the jump; $d_j$ and $u_j$ at the back of the jump and $u_B$ at the downstream boundary. To determine these unknowns we solve the following six equations:

$$\frac{u_{\text{toe}}^2}{2g} + d_{\text{toe}} + \Gamma_{\text{toe}} = \frac{u_j^2}{2g} + d_j + \Gamma_j, \quad u_{\text{toe}}d_{\text{toe}} = Q_0, \quad (59)$$

$$c = \frac{u_{\text{toe}}d_{\text{toe}} - u_jd_j}{d_{\text{toe}} - d_j} = 0,$$

$$c = u_{\text{toe}} - \sqrt{\frac{gd_j}{d_{\text{toe}}} \left( \frac{d_j + d_{\text{toe}}}{2} \right)} = 0, \quad (60)$$

$$\frac{u_B^2}{2g} + d_B = \frac{u_j^2}{2g} + d_j + \Gamma_{\text{toe}}, \quad u_Bd_B = Q_0. \quad (61)$$

For given values of $Q_0$ and $\Gamma_j$, a decrease in $d_B$ will make the jump position move from the vicinity of the bar crest towards the downstream foot of the bar beyond which no steady solutions are possible unless $d_B$ satisfies the super-critical flow conditions.

As a specific case we set up the numerical model (using Scheme 4) covering a channel length of 30 m with the grid size $\Delta x=0.10$ m and the time step $\Delta t=0.03$ s. The submerged bar has a width of $2b=8$ m, and a height of $\Gamma_c=0.2$ m, while the inflow discharge at the western boundary is $Q_0=0.3$ m$^3$/s. The depth $d_B$ at the downstream (eastern) boundary is kept identical to the initial condition $d_0$ which is varied from test to test. As long as $d_0 \geq 0.4953$ m the flow will be subcritical everywhere and the surface elevation will be symmetrical over the bar.

In Fig. 6a we consider the case of $d_0=0.48$ m, which leads to a small growth in the upstream depth to $d_\alpha=0.4953$ m. This involves a small hydraulic jump occurring immediately downstream of the bar crest.

The analytical solution yields $x_{\text{toe}}=16.8$ m, $\Gamma_{\text{toe}}=0.159$ m and a Froude number of $F_{\text{toe}}=1.66$, while the numerical solution yields $x_{\text{toe}}=16.1$ m, $\Gamma_{\text{toe}}=0.185$ m and $F_{\text{toe}}=1.33$. Although this is far from perfect, the overall agreement in Fig. 6a is quite satisfactory.

In Fig. 6b we decrease the initial depth and the eastern boundary level to $d_0=0.37$ m. In this case we get a much stronger hydraulic jump occurring near the downstream foot of the bar. The analytical solution yields $x_{\text{toe}}=19.9$ m, $\Gamma_{\text{toe}}=0.004$ m and $F_{\text{toe}}=2.75$, while the numerical solution yields $x_{\text{toe}}=18.6$ m, $\Gamma_{\text{toe}}=0.038$ m and $F_{\text{toe}}=2.63$. This agreement is most satisfactory and the same conclusion can be made from Fig. 6b.

Finally, when we further decrease $d_B$ a steady jump solution is no longer possible, and the flow remains supercritical all the way to the boundary. Now the outflow conditions have no influence on the upstream flow, and the water depth and velocity maintain their supercritical values from the downstream foot of the bar. This case is shown in Fig. 6c, where the eastern depth becomes $d_B=0.106$ m. The numerical and analytical results are in good agreement.

4. Simulation of the tidal bore on the Qiantang River

South of Shanghai in China, we find the beautiful city of Hangzhou in the province of Zhejiang. This city is connected to the East China Sea via the river of Qiantang Jiang, which runs out into the Hangzhou Bay. This estuary has the shape of a funnel or a trumpet with a dramatic decrease in width as well as in water depth: At the entrance to the sea the outer Hangzhou Bay is about 100 km wide with an average depth of 13 m and at Ganpu, at the entrance to the inner Hangzhou Bay, the width is 22 km with an average depth of 5.5 m. This significantly amplifies the tidal range, which is increased by up to 75% at Ganpu with a maximum historical observation of 9.0 m tidal range. The width is reduced to approximately 9 km near Jianshan and to 4–5 km near Daquekou. In addition, a number of large shoals appear e.g. north of Cao’e River mouth, along the east bank between Jianshan and Daquekou and
along the west bank north of Ershigongduan. They are quite shallow during flood and become dry during ebb tide and this makes the effective width of the cross sections considerably smaller and amplifies the incoming tide.

South of Jianshan and north of Cao’e River Mouth the tidal flood wave develops into the worlds largest tidal bore, the Qiantang Dragon bore. This bore travels with a speed of up to 40 km/h, with a steep front where the surface level jumps up to 3 m within 5 min, and with maximum fluid velocities exceeding 5–6 m/s. After the bore forms it bounces into the southwest banks of Hangzhou Bay and reflects in the northern direction towards Ershigongduan and Daquekou. Further reflections and intersections take place at the river bends at Daquekou and at Lao Yan Can.

Extensive field measurements of the Qiantang bore were made in September 2000, from the Hydraulic Power Station of Fuchun River to the cross-section of Jinstan in Hangzhou Bay. Tidal gauges were set up at Zakou, Qibao, Cangqian, Sigongduan, Yee fluan, Daque-koa, Ershigongduan, Jianshan and at Ganpu (see Fig. 7), and the tidal levels were observed continuously during half a month and recorded every 1 to 2 min. During this period the observed maximum tidal range reached 7.7 m at Ganpu. Bathymetric data are available from a survey in July 2000. A detailed description of the observations can be found in Lin et al. (2002).

To simulate this event, we set up a model covering the Hangzhou Bay and the Qiantang Jiang river with the eastern boundary at Ganpu and the western boundary first at Zakou and later at Cangqian (see Fig. 7). We use 750 times 500 grid points with \( \Delta x=\Delta y=100 \text{ m} \), and 7600 time steps with \( \Delta t=15 \text{ s} \). Numerical simulations are made with MIKE 21 using Scheme 4 with \( \delta=0 \) and \( \beta=0.25 \), and the flow is computed for three spring tides on 16–17 September 2000. We compare the results with Hui and Pan (2003) and Pan et al. (2003), who used a Godunov-type Riemann solver with a flexible grid with a maximum grid size of 820 m, a minimum grid size of 60 m and a time step of 2 s. As Pan et al. (2003) located their western boundary at Cangqian, we also moved our

Fig. 7. The numerical model area covering Qiantang River and Huangzhou Bay.
western boundary to this location. However, it made little difference in the final calibrated model whether we used Cangqian or Zakou as the model boundary.

Fig. 8 shows the boundary conditions at Ganpu (eastern boundary) and at Cangqian (western boundary). The surface elevations are shown relative to the bathymetric datum of Wusong. We notice that the mean water level rises from 2.4 m at Ganpu to approximately 6.5 m at Cangqian. Furthermore, the shape of the time series changes from almost sinusoidal to a very asymmetrical shape with an almost abrupt increase in water level. This makes it almost impossible to use a standard boundary formulation at the western boundary, because even a slight delay in the numerical bore will provoke a strong reflection from this boundary. For this reason we were forced to implement the relaxation technique (described in Section 2.6) using a relaxation zone of 5 km downstream from the western boundary. This gave a significant improvement of the results and very efficiently fixed the problem of spurious reflections.

Several parameters have been varied during the extensive calibration phase which has involved a sensitivity study of boundary conditions, bottom friction and local bathymetric modifications. One of the major problems has been that in almost all computations the numerical bore arrives with a delay compared to the measurements. In order to make the bore move with the correct speed and height, it has been necessary to reduce the friction to a Manning coefficient $M$=300 m$^{1/3}$/s during flood. Admittedly, this friction is exceptionally low, but similar coefficients have been applied by Hui and Pan (2003) and Pan et al. (2003), who used $M$=233 m$^{1/3}$/s during flood, and by Su et al. (2001), who used a Chezy number of 500 m$^{1/2}$/s. A further improvement of the overall phase agreement at all tidal gauges has been obtained by introducing a 10 min phase shift of the eastern boundary data at Ganpu.

The second major calibration problem has been that the surface elevation during ebb tide generally becomes too high upstream of Daquekou. The ebb resistance does have some influence on this level, but here we have faced the paradox than an increase in the ebb friction increases the ebb level in the present model while it decreases the level in the Godunov method.
Fig. 9. a. Computed and measured surface elevation at Yanguan. b. Computed and measured surface elevation at Singongduan.
Fig. 10. a. Computed velocity in the middle of the cross section at Yanguan. b. Computed discharge through cross section at Yanguan.
model of Pan et al. (2003). We have finally settled on an ebb Manning coefficient of $M = 200 \text{ m}^{1/3}/\text{s}$ throughout the model area, while Pan et al. (2003) used some spatial distribution in the range of $M = 80–170 \text{ m}^{1/3}/\text{s}$. This choice, however, still gave us an overestimate of the upstream ebb levels indicating too much friction loss between Cangqian and Daquekou. We have finally reduced these levels to acceptable magnitudes, by increasing the depth locally between Yanguan and Daquekou with 0.5 m, while the height of the sand banks near Daquekou have been reduced by 1 m.

The calibrated model results are presented in Fig. 9a,b, which show the computed and measured surface elevations at Yanguan and Sigongduan. The agreement with the measurements and with the model results of Pan et al. (2003) is excellent. We notice that the bore reaches its maximum at Yanguan, where the surface elevation increases 1.84 m within 1.25 min and 2.65 m within 5 min.

Fig. 10a shows the computed flow velocity at the middle of the cross-section at Yanguan. The maximum velocity during flood reaches $-6 \text{ m/s}$, while
ebb velocities are found to be 2–3 m/s. At the arrival of the bore, the velocity snaps from 1.94 m/s (ebb) to −3.48 m/s (flood) within 5 min. Unfortunately, no observations of velocities are available, so we have compared the computations with Pan et al. (2003). As seen from Fig. 10a, the agreement is excellent during flood while significant differences show up during ebb, where Pan et al. obtain 1–2 m/s.

To further investigate this matter we have determined the temporal variation of the cross-sectional discharge at Yanguan. From Fig. 10b we note that Pan et al. (2003) generally have higher flood discharge and lower ebb discharge compared to the present model. This also leads to a difference in the net-discharge during one tidal cycle, where the present model yields the value 2844 m³/s which corresponds to a net
outgoing velocity of 0.36 m/s, while the model of Pan et al. yields \(-883 \text{ m}^3/\text{s}\), which corresponds to a net ingoing velocity of \(-0.11 \text{ m/s}\). While a net ingoing velocity is obviously not correct, it remains an open question whether our outgoing velocity has the correct magnitude or not. Additional field measurements of ebb velocities would clarify this matter. We check the net discharges at other cross sections and find the values 2807 m$^3$/s, 3069 m$^3$/s, 3146 m$^3$/s and 3014 m$^3$/s at Sigongduan, Daquekou, Jianshan and Ganpu, respectively. For practical purposes these values are reasonably constant and at least consistent with respect to net outgoing flow providing some confidence to the model. Similar values based on the model by Pan et al. (2003) show much larger variation and no consistent direction of the net flow.

Fig. 11a,b shows two-dimensional close ups of the tidal bore in the region between Jianshan and Daquekou. The bore front is very clear and the fluid velocity changes from +2 m/s ebb to \(-6 \text{ m/s}\) flood over a relatively short distance. In Fig. 11b we notice that the velocities along the western convex bank are significantly larger than along the eastern concave bank. This is in agreement with earlier observations by Pan et al. (2003) and has also been verified in a physical model. More details on this study can be found in the report by Simonsen (2003).

5. Summary and conclusions

The objective of this work has been to modify a well established implicit finite difference formulation of the nonlinear shallow water equations (the S21, Jupiter scheme of Abbott et al., 1973) in order to allow for the modelling of tidal bores and hydraulic jumps. In Section 2, we have investigated various techniques such as upwind treatment of convective terms (Scheme 1), forward time-centering of spatial derivatives in the mass equation (Scheme 2), dissipative interface (Scheme 3), and a combination of forward-time centering and dissipative interface (Scheme 4). The dissipative features of these schemes as a function of the Courant number \((Cr)\) have been analyzed on the basis of the effective amplification portrait, which we define as the magnitude of the maximum eigenvalue (of the amplification matrix) raised to the power of \(1/Cr\). We conclude that the dissipative interface has little effect for \(Cr>1\), while forward time-centering has little effect for \(Cr<1\). Hence, in order to develop a scheme which is efficient in the range of \(0.25\leq Cr\leq 2.0\), we have combined these two features in our Scheme 4.

Various canonical test cases have been investigated in Section 3 including the propagation of a periodic bore, the propagation and reflection of a uniform bore, and the formation of steady hydraulic jumps in connection with flow over a submerged bar. Numerical results with five different schemes have been compared with analytical solutions and with a Godunov Riemann solver. We conclude that Scheme 4 is superior to the other schemes, and that it performs very well in the interval of \(0.25\leq Cr\leq 1.5\). It is also concluded that Scheme 1 (using upwind representation of convective terms) is surprisingly inaccurate unless \(Cr<0.2\).

In Section 4, we make a case study of the fascinating tidal bore in Huangzhou Bay and Qiantang River. After an extensive calibration phase involving a sensitivity study of boundary conditions, bottom friction, and local bathymetric modifications we have succeeded in modelling this complicated phenomenon. The computed surface elevations compare very well with field measurements, and the flow pattern confirms special observed features such as the convex bank velocity being significantly larger than the concave bank velocity near Daquekou. Unfortunately, no velocity measurements are available and we recommend that a future field campaign would concentrate on such measurements at least during ebb tide. Also it would be most interesting to get some information about net currents in Qiantang River.

Finally, we want to emphasize that the developed model using the recommended Scheme 4, in some respect is far less sophisticated than many modern developments based on Godunov Riemann solvers. It is, however, extremely efficient partly because of the relatively high Courant numbers which can be used and partly because of the efficient ADI algorithm. As an example the Huangzhou model consists of 375,000 grid points of which 55,000 are water points. The 32 h tidal event from September 2000 is covered by 7600 time steps of 15 s. The CPU time for this simulation is no more than 55 min on a 2.66 GHz Dell Pc.
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